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Multiply warped products

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Abstract

In this paper, we first state covariant derivative formulas for multiply warped products and consider the geodesic equations for these spaces. Then we state some basic facts about causality of Lorentzian multiply products and study Cauchy surfaces and global hyperbolicity. Finally, we consider null, time-like and space-like geodesic completeness of Lorentzian multiply products and geodesic completeness of Riemannian multiply warped products. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

O'Neill and Bishop [7] introduced *singly warped products* or simply *warped products* to construct Riemannian manifolds with negative sectional curvature. Later, it was pointed out that many exact solutions to *Einstein's field equation* can be expressed in terms of *Lorentzian warped products* by Beem et al. [4]. Moreover, Beem and Ehrlich [3] proved that causality and completeness of warped products can be related to causality and completeness of components of warped products. *Curvature* formulas of singly warped products in terms of curvatures of components of warped products were explored by O'Neill [13] and he also examined *Robertson–Walker, static, Schwarschild* and *Kruskal space–times* as warped products. Also, warped products were considered as *Riemannian submersions* by Besse [6] and he obtained some results for special cases.

In the present work, we study *multiply warped products* or *multiwarped products*. Before we see a brief definition of multiply warped products, we describe the following type of products. A Lorentzian warped product (M, g) of the form $M = (c, d) \times {}_bF$ with the

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metric $g = -dt^2 \oplus b^2 g_F$ is a *generalized Robertson–Walker* space–time, where $-\infty \le c < d \le \infty$ and b: $(c, d) \to (0, \infty)$ is a smooth function. Generalized Robertson–Walker space–times are considered as model space–times in relativity theory (cf. [3,11,13,21]). Especially, in [18] some results about stability, geodesic completeness and geodesic connectedness of generalized Robertson–Walker space–times were stated. Furthermore, in [21] necessary and sufficient conditions for a generalized Robertson–Walker space–time to have positive Ricci curvature on non-space-like tangent vectors and some conditions for them to be either Ricci-flat or Einstein are proven. Also, in [21] some results for a generalized Robertson–Walker space–time to have non-negative sectional curvature on time-like plane sections are established and some applications to singularity theorems are given and also some results for certain types of warped products to have constant scalar curvature are stated.

In general, *doubly warped products* can be considered as generalizations of singly warped products. A doubly warped product (M, g) is a product manifold which is of the form $M = {}_f B \times {}_b F$ with the metric $g = f^2 g_B \oplus b^2 g_F$ where b: $B \to (0, \infty)$ and f: $F \to (0, \infty)$ are smooth maps. Beem and Powell [5] considered these products for Lorentzian manifolds. Then Allison [1] considered *causality* and *global hyperbolicity* of doubly warped products. In [22], Cauchy surfaces in doubly warped products and global hyperbolicity are considered. Then geodesic completeness of Lorentzian doubly warped products and Riemannian doubly warped products are studied and necessary conditions are given for generalized Robertson–Walker space–times with doubly warped product spacial parts to be globally hyperbolic. Also a k-decaying condition is used to establish some results about geodesic incompleteness of Riemannian doubly warped products, in addition to those, some results are stated about Killing and conformal vector fields of doubly warped products.

One can also generalize singly warped products to multiply warped products. Briefly, a multiply warped product (M, g) is a product manifold of the form $M = B \times_{b_1} F_1 \times_{b_2} F_2 \times \cdots \times_{b_m} F_m$ with the metric $g = g_B \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \oplus \cdots \oplus b_m^2 g_{F_m}$, where for each $i \in \{1, \ldots, m\}$, b_i: $B \to (0, \infty)$ is smooth and (F_i, g_{F_i}) is a *pseudo-Riemannian* manifold. *Covariant derivatives* and curvatures of multiply warped products are given in [2] for $m = 2$. In particular, when $B = (c, d)$ with the negative definite metric $g_B = -dt^2$, the corresponding multiply warped product $M = (c, d) \times_{b_1} F_1 \times_{b_2} F_2 \times \cdots \times_{b_m} F_m$ with the metric $g = -dt^2 \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \oplus \cdots \oplus b_m^2 g_{F_m}$ is called a multiply warped space–time, where for each $i \in \{1, \ldots, m\}$, (F_i, g_{F_i}) is a Riemannian manifold and $-\infty \le c < d \le \infty$. Geodesic equations and geodesic connectedness of multiply warped space–times were studied by Flores and Sánchez [9] and they also noted that the class of multiply warped space–times contains many well-known relativistic space–times.

There are various types of warped products in addition to the ones considered above and some of these have proven useful in general relativity. Campbell [8] studied local embeddings of pseudo-Riemannian manifolds in Ricci-flat pseudo-Riemannian manifolds. His work was used to construct the local embedding in five-dimensional, Ricci-flat spaces of four-dimensional space–times admitting a non-twisting, null *Killing* vector field in [12] and to show that general relativistic solutions can always be locally embedded in Ricci-flat five-dimensional spaces in [17]. It can be easily observed that all these extensions are some mixed types of warped products. Also, in [15] some physically motivated D-dimensional solutions studied by Wesson and Ponce de Leon were extended to $(D+1)$ dimensions and again these extensions turn out to be various types of warped products.

In Section 3, we study the *causal* structures of multiply warped products. Then we also investigate *Cauchy surface* structures and *global hyperbolicity* of multiply warped products and obtain generalizations of singly warped products results (cf. [3]).

In Section 4, we consider geodesic completeness of Lorentzian and Riemannian doubly warped products. We examine *null geodesic completeness* of Lorentzian multiply warped products (M, g) of the form $M = (c, d) \times {}_{b_1} F_1 \times \cdots \times {}_{b_m} F_m$ with metric $g = -dt^2 \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$, where $-\infty \le c < d \le \infty$ and $\mathcal{B} = \{b_1, \ldots, b_m\}$ by using similar techniques in [16]. To do this we suppose that (F_i, g_{F_i}) is a complete Riemannian manifold for any $i \in \{1, \ldots, m\}$ then we get relations between null geodesic completeness of (M, g) and the divergence of $\lim_{t\to c^+} \int_t^{w_0} (f[\bar{b}_1, \ldots, \bar{b}_k](s)/\sqrt{h[\bar{b}_1, \ldots, \bar{b}_k](s)})\, \mathrm{d}s$ and $\lim_{t\to d^-} \int_{w_0}^t (f[\bar{b}_1, \ldots, \bar{b}_k](s)/\sqrt{h[\bar{b}_1, \ldots, \bar{b}_k](s)})\, \mathrm{d}s$ for some $w_0 \in (c, d)$ and any $k \in \{1, \ldots, m\}$ and any subset $\{\bar{b}_1, \ldots, \bar{b}_k\}$ of $\mathcal{B}$, where

$$f[\bar{b}_1, \ldots, \bar{b}_k] = \prod_{i=1}^{k} \bar{b}_i \quad \text{and} \quad h[\bar{b}_1, \ldots, \bar{b}_k] = \sum_{i=1}^{k} \bar{b}_1^2 \cdots \bar{b}_{i-1}^2 \bar{b}_{i+1}^2 \cdots \bar{b}_k^2.$$

Similarly, we also examine *time-like geodesic completeness* of (M, g) and we obtain relations between time-like geodesic completeness of (M, g) and the divergence of $\lim_{t\to c^+} \int_t^{w_0} (f[\bar{b}_1, \ldots, \bar{b}_k](s)/\sqrt{f[\bar{b}_1, \ldots, \bar{b}_k]^2(s) + h[\bar{b}_1, \ldots, \bar{b}_k](s)})\, \mathrm{d}s$ and $\lim_{t\to d^-} \int_{w_0}^t (f[\bar{b}_1, \ldots, \bar{b}_k](s)/\sqrt{f[\bar{b}_1, \ldots, \bar{b}_k]^2(s) + h[\bar{b}_1, \ldots, \bar{b}_k](s)})\mathrm{d}s$ for more some $w_0 \in (c, d)$ and any $k \in \{1, \ldots, m\}$ and any subset $\{\bar{b}_1, \ldots, \bar{b}_k\}$ of $\mathcal{B}$. Finally, we consider space-like geodesic completeness of (M, g) and obtain relations between space-like geodesic completeness (M, g) and the divergence of $\lim_{t\to c^+} \int_t^{w_0} (f[\bar{b}_1, \ldots, \bar{b}_k](s)/\sqrt{h[\bar{b}_1, \ldots, \bar{b}_k](s)})\, \mathrm{d}s$ and $\lim_{t\to d^-} \int_{w_0}^t (f[\bar{b}_1, \ldots, \bar{b}_k](s)/\sqrt{h[\bar{b}_1, \ldots, \bar{b}_k](s)})\, \mathrm{d}s$ or unboundedness of $f[\bar{b}_1, \ldots, \bar{b}_k](s)/\sqrt{h[\bar{b}_1, \ldots, \bar{b}_k](s)}$ on (w_0, d) or (c, w_0) for some $w_0 \in (c, d)$ and any $k \in \{1, \ldots, m\}$ and any subset $\{\bar{b}_1, \ldots, \bar{b}_k\}$ of $\mathcal{B}$.

Moreover, we extend some results about geodesic completeness of Lorentzian multiply warped products from [14], i.e., when a Lorentzian multiply warped product (M, g) of the form $M = B \times {}_{b_1} F_1 \times \cdots \times {}_{b_m} F_m$ with metric $g = g_B \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$ is null, time-like or space-like complete then (F_i, g_{F_i}) is a complete Riemannian manifold for any $i \in \{1, \ldots, m\}$ and in this case, (B, g_B) is null, time-like or space-like complete, respectively.

After considering geodesic completeness of Lorentzian multiply warped products we turn our attention to Riemannian multiply warped products (M, g) of the form $M = B \times {}_{b_1} F_1 \times \cdots \times {}_{b_m} F_m$ with metric $g = g_B \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$. We proved that if (B, g_B) and (F_i, g_{F_i}) are all complete Riemannian manifolds for any $i \in \{1, \ldots, m\}$, then (M, g) is also complete and conversely, when (M, g) is complete then (B, g_B) and (F_i, g_{F_i}) are complete for any $i \in \{1, \ldots, m\}$.

2. Preliminaries

Thoughout this work any manifold M is assumed to be connected, Hausdorff, paracompact and smooth. A pseudo-Riemannian manifold (M, g) is a smooth manifold with a metric tensor g and a Lorentzian manifold (M, g) is a pseudo-Riemannian manifold with signature $(-, +, +, \dots, +)$.

Let (M, g) be a Lorentzian manifold. A non-zero tangent vector $X_p \in T_p(M)$ is said to be time-like (respectively, space-like or null) if $g(X_p, X_p) < 0$ (respectively, $g(X_p, X_p) > 0$ or $g(X_p, X_p) = 0$).

A Lorentzian manifold (M, g) is called *time-oriented* by the vector field X, if X is time-like at every point of M. A time-oriented Lorentzian manifold (M, g) is called a space–time.

Let $p, q \in M$. Then $p \ll q$ if there exists a smooth future directed time-like curve from p to q and $p \leq q$ if there exists a smooth future directed non-space-like curve from p to q. The chronological future $I^+(p)$ of p is the set $I^+(p) = \{q \in M | p \ll q\}$ and the chronological past $I^-(p) = \{q \in M | q \ll p\}$. The causal future $J^+(p)$ of p is the set $J^+(p) = \{q \in M | p \leq q\}$ and the causal past $J^-(p) = \{q \in M | q \leq p\}$.

Now, we briefly state some causality conditions in order of increasing strength (cf. [3,11]). If a space–time (M, g) contains no closed time-like curves then (M, g) is chronological. A space–time with no closed non-space-like curves is called causal. An open set U in a space–time is called causally convex if no non-space-like curve intersects U in a disconnected set. Given $p \in M$, the space–time (M, g) is called strongly causal at p if p has arbitrarily small causally convex neighborhoods. A space–time is said to be a strongly causal space–time if it is strongly causal at each point.

A space–time (M, g) is stably causal if there is a fine C^0 neighborhood $U(g)$ of g in Lor (M) such that each $g_1 \in U(g)$ is causal. A continuous function $f\colon M \to \mathbb{R}$ is a global time function if f is strictly increasing along each future directed time-like curve. A space–time is stably causal if and only if it has a global time function. A strongly causal space–time (M, g) is said to be globally hyperbolic if for each pair of points $p, q \in M$ the set $J^+(p) \cap J^-(q)$ is compact. Globally hyperbolic space–times may be characterized by using Cauchy surfaces. A subset of M which every inextendible non-space-like curve intersects exactly once is called a Cauchy surface. A space–time is globally hyperbolic if and only if it has a Cauchy surface (cf. [10,11]). At this point we recall that in a globally hyperbolic space–time any pair of causally related points may be joined by a non-space-like geodesic segment of maximal length (cf. [19]).

Let (M, g) be a Lorentzian manifold. Given $p, q \in M$, with $p \leq q$, define $\Omega_{p,q}$ as the set of all future directed piecewise smooth non-space-like curves $\gamma\colon [0, 1] \to M$ from p to q, i.e., $\gamma(0) = p$ and $\gamma(1) = q$. The *Lorentzian distance* $d\colon M \times M \to \mathbb{R} \cup \{\infty\}$ is defined as follows: let $p, q \in M$ then

$$d(p, q) = \begin{cases} 0 & \text{if } q \notin J^+(p), \\ \sup\{L_g(\gamma) | \gamma \in \Omega_{p,q} & \text{if } q \in J^+(p), \end{cases}$$

where $L_g(\gamma)$ is the *Lorentzian arc length* of γ.

In arbitrary Lorentzian manifolds, a reverse triangle inequality holds. More explicitly, if $p \leq q \leq r$ then $d(p,q) + d(q,r) \leq d(p,r)$. In globally hyperbolic space–times, the Lorentzian distance function is finite and continuous.

A smooth curve $\gamma\colon I \to M$ in an arbitrary pseudo-Riemannian manifold is said to be a pre-geodesic if it can be reparametrized so that the repametriziation is a geodesic. A parameter s for a pre-geodesic γ is called an affine parameter if $\gamma''(s) = 0$.

A smooth curve $\gamma\colon (a,b) \to M$ in an arbitrary pseudo-Riemannian manifold is inextendible to $t = b$ (respectively, to $t = a$) if the $\lim_{t\to b^-}\gamma(t)$ (respectively, $\lim_{t\to a^+}\gamma(t)$) does not exist.

In a Riemannian (i.e., positive definite) manifold the Hopf–Rinow theorem (cf. [13]) states the equivalence of metric completeness and geodesic completeness.

The Lorentzian manifold (M, g) is time-like (respectively, null, space-like) complete if all time-like (respectively, null, space-like) inextendible geodesics are complete (i.e., can be defined on all of $\mathbb{R}$). A non-space-like incomplete space–time is called a geodesically singular space–time (cf. [10]).

Here, we briefly explain the topology of warped products.

Let (B, g_B) and (F_i, g_{F_i}) be r and s_i dimensional pseudo-Riemannian manifolds, respectively, where $i \in \{1, 2, \ldots, m\}$. If $F = F_1 \times F_2 \times \cdots \times F_m$, then $M = B \times F$ is an n-dimensional pseudo-Riemannian manifold where $s = \sum_{i=1}^{m} s_i$ and $n = r + s$.

Throughout this paper we use the *natural product coordinate system* on the product manifold $B \times F$. Let $(p, q_1, q_2, \ldots, q_m)$ be a point in M. Then there are *coordinate charts* (U, x) and (V_i, y_i) on B and F_i, respectively, where $i \in \{1, 2, \ldots, m\}$ such that $p \in B$ and $q_i \in F_i$. Then we can define a coordinate chart (W, z) on M such that W is an open subset in M contained in $U \times V_1 \times V_2 \times \cdots \times V_m$ and $(p, q_1, q_2, \ldots, q_m) \in W$ then for all $(u, v_1, v_2, \ldots, v_m)$ in $W, z(u,v) = (x(u), y_1(v_1)y_2(v_2), \ldots, y_m(v_m))$, where $\pi\colon B \times F \to B$ and $\sigma_i\colon B \times F \to F_i$ and also $\sigma\colon B \times F \to F$ are usual projection maps where $i \in \{1, 2, \ldots, m\}$. Clearly, the set of all (W, z) defines an atlas on $B \times F$.

Let $\phi\colon B \to \mathbb{R} \in \mathcal{D}(B)$ then the lift of ϕ to $B \times F$ is $\tilde{\phi} = \phi \circ \pi \in \mathcal{D}(B \times F)$, where $\mathcal{D}(B)$ is the set of all smooth real-valued functions on B.

Moreover, one can define *lifts* of vector fields as: let $X \in \mathcal{X}(B)$ then the lift of X to $B \times F$ is the vector field $\tilde{X} \in \mathcal{X}(B \times F)$ such that $d\pi(\tilde{X}) = X$ and $d\sigma_i(\tilde{X}) = 0$ for any $i \in \{1, 2, \ldots, m\}$. Similarly, let $V_i \in \mathcal{X}(F_i)$ then the lift of V_i to $B \times F$ is the vector field $\tilde{V_i} \in \mathcal{X}(B \times F)$ such that $d\pi(\tilde{V_i}) = 0$ and $d\sigma_i(\tilde{V_i}) = V_i$ and also $d\sigma_j(\tilde{V_i}) = 0$ for any $j \in \{1, 2, \ldots, m\} - \{i\}$. We will denote the set of all lifts of all vector fields of B by $\mathcal{L}(B)$ and the set of all lifts of all vector fields of F_i by $\mathcal{L}(F_i)$ for any $i \in \{1, 2, \ldots, m\}$. Now we are ready to define multiply warped products.

Definition 2.1. Let (B, g_B) and (F_i, g_{F_i}) be pseudo-Riemannian manifolds and also let $b_i\colon B \to (0, \infty)$ be smooth functions for any $i \in \{1, 2, \ldots, m\}$. The multiply warped product is the product manifold $B \times F_1 \times F_2 \times \cdots \times F_m$ furnished with the metric tensor $g = g_B \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \oplus \cdots \oplus b_m^2 g_{F_m}$ defined by

$$g = \pi^*(g_B) \oplus (b_i \circ \pi)^2 \sigma_1^*(g_{F_1}) \oplus \cdots \oplus (b_m \circ \pi)^2 \sigma_m^*(g_{F_m}). \tag{1}$$

The functions $b_i\colon B \to (0,\infty)$ are called warping functions for any $i \in \{1, 2, \dots, m\}$. If $m = 1$, then we obtain a singly warped product. If all $b_i \equiv 1$, then we have a product manifold. If (B, g_B) and (F_i, g_{F_i}) are all Riemannian manifolds, then $(B \times_{b_1} F_1 \times_{b_2} F_2 \times \cdots \times_{b_m} F_m, g_B \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \oplus \cdots \oplus b_m^2 g_{F_m})$ is also a Riemannian manifold. We call $(B \times_{b_1} F_1 \times_{b_2} F_2 \times \cdots \times_{b_m} F_m, g_B \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \oplus \cdots \oplus b_m^2 g_{F_m})$ Lorentzian doubly warped product if (F_i, g_{F_i}) are all Riemannian and either (B, g_B) is Lorentzian or else (B, g_B) is a one-dimensional manifold with a *negative definite* metric $-dt^2$.

In [2], *metric components, covariant derivatives, Riemannian curvature, Ricci curvature* and *scalar curvatures* of multiply warped products are studied for $m = 2$. We will state the covariant derivative formulas for multiply warped products. Note that the these formulas are proven in [2] for $m = 2$.

Proposition 2.2. *Let $M = B \times_{b_1} F_1 \times \cdots \times_{b_m} F_m$ be a pseudo-Riemannian multiply warped product with metric $g = g_B \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$ also let $X, Y \in \mathcal{L}(B)$ and $V \in \mathcal{L}(F_i)$, $W \in \mathcal{L}(F_j)$. Then*

1. $\nabla_X Y = \widetilde{(\nabla_X^B Y)}$,
2. $\nabla_X V = \nabla_V X = \dfrac{X(b_i)}{b_i} V$,
3. $\nabla_V W = \begin{cases} 0 & if \;\; i \neq j, \\ \widetilde{(\nabla_V^{F_i} W)} - (g(V, W)/b_i)\,\mathrm{grad}_B(b_i) & if \;\; i = j. \end{cases}$

By using the above result, it is easy to obtain the following generalizations of results [13] for singly warped products.

Proposition 2.3. *Let $M = B \times_{b_1} F_1 \times \cdots \times_{b_m} F_m$ be a pseudo-Riemannian multiply warped product with metric $g = g_B \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$. Then*

1. *The leaves $B \times \{q\}$ and the fibers $\{p\} \times F$ of the multiply warped product are totally umbilic.*
2. *The leaf $B \times \{q\}$ is totally geodesic, and the fiber $\{p\} \times F$ is totally geodesic if $\mathrm{grad}_B(b_i)|_p = 0$ for any $i \in \{1, 2, \dots, m\}$.*

Now, we will state the geodesic equations for multiply warped products. The version of this fact for singly warped products and doubly warped products are well known (cf. [13,21]).

Proposition 2.4. *Let $M = B \times_{b_1} F_1 \times \cdots \times_{b_m} F_m$ be a pseudo-Riemannian multiply warped product with metric $g = g_B \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$. Also let $\gamma = (\alpha, \beta_1, \dots, \beta_m)$ be a curve in M defined on some interval $I \subseteq \mathbb{R}$. Then $\gamma = (\alpha, \beta_1, \dots, \beta_m)$ is a geodesic in M if and only if for any $t \in I$,*

1. $\alpha'' = \displaystyle\sum_{i=1}^{m} (b_i \circ \alpha) g_{F_i}(\beta_i', \beta_i')\,\mathrm{grad}_B(b_i)$.

2. $\beta_i'' = \dfrac{-2}{(b_i \circ \alpha)} \dfrac{\mathrm{d}(b_i \circ \alpha)}{\mathrm{d}t} \beta_i'$, *for any* $i \in \{1, 2, \ldots, m\}$.

Remark 2.5. *Let* $M = B \times_{b_1} F_1 \times \ldots \times_{b_m} F_m$ *be a pseudo-Riemannian multiply warped product with metric* $g = g_B \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$. *Also let* $\gamma = (\alpha, \beta_1, \ldots, \beta_m)$ *be a curve in M defined on some interval* $I \subseteq \mathbb{R}$. *If* $\gamma = (\alpha, \beta_1, \ldots, \beta_m)$ *is a geodesic in M, then*

1. $\beta_i \colon I \to F_i$ *is a pre-geodesic in* F_i *for any* $i \in \{1, 2, \ldots, m\}$.
2. $(b_i \circ \alpha)^4 g_{F_i}(\beta_i', \beta_i') \equiv c_i$ *for any* $i \in \{1, 2, \ldots, m\}$.
3. α *is a constant if and only if there exists a point* $s \in I$ *such that* $\alpha'(s) = 0$ *and* $c_i = 0$ *for any* $i \in \{1, 2, \ldots, m\}$ or $\alpha'(s) = 0$ and $\mathrm{grad}_B(b_i)(\alpha(s)) = 0$ *for any* $i \in \{1, \ldots, m\}$.
4. β_i *is constant for some* $i \in \{1, \ldots, m\}$ *if and only if* $c_i = 0$.

3. Causality of multiply warped products

In this section, we briefly recall *causal structures of multiply warped products* and state some results about *global hyperbolicity* of multiply warped products. All the results can be proven by using the similar arguments to prove the analogues of these results for singly warped products (cf. [3]).

3.1. Causality

In this section, we will generalize some basic facts about causality of Lorentzian singly warped products to Lorentzian multiply warped products (cf. [3]).

Let $M = B \times_{b_1} F_1 \times \cdots \times_{b_m} F_m$ be a Lorentzian multiply warped product with metric $g = g_B \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$. Then we have,

1. If $(p, q) \in M$ then $d\pi_{(p,q)} \colon T_{(p,q)}(B \times F) \to T_p(B)$ maps non-space-like vectors of $T_{(p,q)}(B \times F)$ to non-space-like vectors of $T_p(B)$ and $\pi \colon B \times F \to B$ maps non-space-like curves of $B \times F$ to non-space-like curves of B.
2. The map $\pi \colon B \times F \to B$ is length non-decreasing on non-space-like curves of $B \times F$.
3. (M, g) is time-orientable if and only if (B, g_B) is time-orientable (if $r \geq 2$) or (B, g_B) is a one-dimensional manifold with a negative definite metric.
4. If q is a point in F then each leave $\sigma^{-1}(q) = B \times \{q\}$ has the same chronology and causality as (B, g_B).

Suppose that (p_1, q_1), (p_2, q_2), (p_1, q), and (p_2, q) are points in M then

1. if $(p_1, q_1) \ll (p_2, q_2)$ then $p_1 \ll p_2$,
2. if $(p_1, q_1) \leq (p_2, q_2)$ then $p_1 \leq p_2$,
3. if $p_1 \ll p_2$ then $(p_1, q) \ll (p_2, q)$,
4. if $p_1 \leq p_2$ then $(p_1, q) \leq (p_2, q)$.

By using the similar arguments in [3], we get the following.

Theorem 3.1. *Let* $M = (c, d) \times_{b_1} F_1 \times \cdots \times_{b_m} F_m$ *be a Lorentzian multiply warped product with metric* $g = -dt^2 \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$, *where* $-\infty \leq c < d \leq \infty$. *Then* (M, g)

is stably causal and consequently strongly causal, distinguishing, causal and chronological.

Theorem 3.2. *Let $M = B \times_{b_1} F_1 \times \cdots \times_{b_m} F_m$ be a Lorentzian multiply warped product with metric $g = g_B \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$. Then*

1. *(M, g) is causal (respectively, chronological) if and only if the space–time (B, g_B) is causal (respectively, chronological).*
2. *(M, g) is strongly causal (respectively, stably causal) if and only if the space–time (B, g_B) is strongly causal (respectively, stably causal).*

Note that if B is diffeomorphic to $\mathbb{S}^1$, then $\mathbb{S}^1 \times_{b_1} F_1 \times \cdots \times_{b_m} F_m$ is never chronological.

3.2. Global hyperbolicity

In this section, we will generalize some basic facts about global hyperbolicity of Lorentzian singly warped product to Lorentzian multiply warped products (cf. [3]).

Theorem 3.3. *Let $M = B \times_{b_1} F_1 \times \cdots \times_{b_m} F_m$ be a Lorentzian multiply warped product with metric $g = g_B \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$. Then (M, g) is globally hyperbolic if and only if*

1. *(B, g_B) is globally hyperbolic and*
2. *(F_i, g_{F_i}) is complete for any $i \in \{1, \ldots, m\}$.*

Corollary 3.4. *Let $M = (c, d) \times_{b_1} F_1 \times \cdots \times_{b_m} F_m$ is a Lorentzian multiply warped product with metric $g = -dt^2 \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$,* where $-\infty \leq c < d \leq \infty$. *Then (M, g) is globally hyperbolic if and only if (F_i, g_{F_i}) is complete for any $i \in \{1, \ldots, m\}$.*

Now, we will give the following result about Cauchy surfaces in multiply warped products.

Theorem 3.5. *Let $M = B \times_{b_1} F_1 \times \cdots \times_{b_m} F_m$ be a Lorentzian multiply warped product with metric $g = g_B \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$. If (F_i, g_{F_i}) is complete for any $i \in \{1, \ldots, m\}$, then*

1. *if $B = (c, d)$* for $-\infty \leq c < d \leq \infty$ *is given the negative definite metric $-dt^2$* then *$\{p\} \times F$ is a Cauchy surface of (M, g) for every $p \in B$,*
2. *if (B, g_B) is a globally hyperbolic space–time with a Cauchy surface S_B then $S_B \times F$ is a Cauchy surface of (M, g).*

4. Completeness of multiply warped products

In this section, we obtain some results about geodesic completeness of Lorentzian and Riemannian warped products. Analogues of these results for both Lorentzian and Riemannian singly and doubly warped products are studied in [3,14,16,22].

4.1. Lorentzian warped products

In this section, we state some results about null and time-like geodesic completeness of Lorentzian multiply warped products.

We now consider the non-space-like geodesic completeness of Lorentzian multiply warped products of the form $M = (c, d) \times {}_{b_1} F_1 \times \cdots \times {}_{b_m} F_m$ with metric $g = -dt^2 \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$, where $-\infty \leq c < d \leq \infty$. Here, a space–time is said to be null (respectively, time-like) geodesically incomplete if at least one future directed null (respectively, time-like) geodesic cannot be extended to be defined or arbitrary negative and positive values of an affine parameter. Since we are using the metric $-dt^2$ on (c, d), the curve $\gamma(t) = (t, q)$ with $q \in F$ fixed and is a unit speed time-like geodesic (M, g) independent of which warping functions $b_1, \ldots, b_m$ are chosen.

Consequently, if $c > -\infty$ or $d < \infty$ then (M, g) is time-like geodesically incompletely for all possible functions warping functions $b_1, \ldots, b_m$. Moreover, if c and d are both finite and if γ is any time-like geodesic in M, then $L(\gamma) \leq d - c < \infty$. Thus if c and d are finite, all time-like geodesics are past and future incomplete (cf. [3]).

Lemma 4.1. *$M = (c, d) \times {}_{b_1} F_1 \times \cdots \times {}_{b_m} F_m$ be a Lorentzian multiply warped product with metric $g = -dt^2 \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$, where $-\infty \leq c < d \leq \infty$.* Also let *$\gamma = (\alpha, \beta_1, \ldots, \beta_m)\colon I \to M$ be a geodesic in M. If there exist $i \in \{1, \ldots, m\}$ and points $q_j \in F_j$ for all $j \in \{1, \ldots, m\} - \{i\}$ such that $\beta_j(t) = q_j$, for any $t \in I$, and $\beta_j'(0) = 0$, for all $j \in \{1, \ldots, m\} - \{i\}$, then $\beta_j'(t) = 0$ for all $j \in \{1, \ldots, m\} - \{i\}$ and for any $t \in I$.*

Proof. If γ is a geodesic in M, then $\beta_k'' = -2/(b_k \circ \alpha)\mathrm{d}(b_k \circ \alpha)/\mathrm{d}t\, \beta_k'$ for any $k \in \{1, \ldots, m\}$ by Proposition 2.4. If $j \in \{1, \ldots, m\} - \{i\}$, then $\beta_j''(0) = 0$, $\beta_j'(0) = 0$ and hence $\beta_j'(t) \equiv 0$ satisfies both equations. Thus by the existence and the uniqueness of solutions of ordinary differential equations, we have that $\beta_j'(t) \equiv 0$ for all $j \in \{1, \ldots, m\} - \{i\}$. □

Clearly, if $\gamma = (\alpha, \beta_1, \ldots, \beta_m)\colon I \to M$ is a null (respectively, time-like or space-like) geodesic in M such that there exist $i \in \{1, \ldots, m\}$ and points $q_j \in F_j$, for all $j \in \{1, \ldots, m\} - \{i\}$ with $\beta_j(t) = q_j$, for any $t \in I$, then it follows from Lemma 4.1 that γ is null (respectively, time-like or space-like) incomplete in (M, g) if and only if (α, β_i) is null (respectively, time-like or space-like) incomplete in $((c, d) \times {}_{b_i} F_i, -dt^2 \oplus b_i^2 g_{F_i})$. Using Lemma 4.1 and techniques for singly warped products (cf. [3,16,18]) we may establish the following three results.

Theorem 4.2. *$M = (c, d) \times {}_{b_1} F_1 \times \cdots \times {}_{b_m} F_m$ be a Lorentzian multiply warped product with metric $g = -dt^2 \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g F_m$, where $-\infty \leq c < d \leq \infty$. Then*

1. *if $\lim_{t \to c^+} \int_t^{w_0} b_i(s)\, \mathrm{d}s < \infty$ for some $w_0 \in (c, d)$ and for some $i \in \{1, \ldots, m\}$ then some future directed null geodesics are past incomplete and thus (M, g) is future directed null geodesic past incomplete,*
2. *if $\lim_{t \to d^-} \int_{w_0}^t b_i(s)\, \mathrm{d}s < \infty$ for some future for some $w_0 \in (c, d)$ and for some $i \in \{1, \ldots, m\}$ then some future directed null geodesics are future incomplete and thus (M, g) is future directed null geodesic future incomplete.*

Theorem 4.3. *$M = (c, d) \times_{b_1} F_1 \times \cdots \times_{b_m} F_m$ be a Lorentzian multiply warped product with metric $g = -dt^2 \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$, where $-\infty \leq c < d \leq \infty$. Then*

1. *if $\lim_{t\to c^+} \int_t^{w_0} b_i(s)/\sqrt{1+b_i^2(s)}\, \mathrm{d}s < \infty$ for some $w_0 \in (c, d)$ and for some $i \in \{1, \ldots, m\}$ then some future directed time-like geodesic is past incomplete and thus (M, g) is future directed time-like geodesic past incomplete,*
2. *if $\lim_{t\to d^-} \int_{w_0}^t b_i(s)/\sqrt{1+b_i^2(s)}\, \mathrm{d}s < \infty$ for some $w_0 \in (c, d)$ and for some $i \in \{1, \ldots, m\}$ then some future directed time-like geodesic is future incomplete and thus (M, g) is future directed time-like geodesic future incomplete.*

Theorem 4.4. *$M = (c, d) \times_{b_1} F_1 \times \cdots \times_{b_m} F_m$ be a Lorentzian multiply warped product with metric $g = -dt^2 \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$, where $-\infty \leq c < d \leq \infty$. Then*
1. *if $\lim_{t\to c^+} \int_t^{w_0} b_i(s)\, \mathrm{d}s < \infty$* and *$b_i$ is a bounded function on (c, w_0) for some $w_0 \in (c, d)$ and for some $i \in \{1, \ldots, m\}$ then some future directed space-like geodesic is past incomplete and thus (M, g) is future directed space-like geodesic past incomplete,*
2. *if $\lim_{t\to d^-} \int_{w_0}^t b_i(s)\, \mathrm{d}s < \infty$ and b_i is a bounded function on (w_0, d) for some $w_0 \in (c, d)$ and for some $i \in \{1, \ldots, m\}$ then some future directed space-like geodesic is future incomplete and thus (M, g) is future directed space-like geodesic future incomplete.*

Now, we will obtain some integral conditions to guarantee null, time-like and space-like geodesic completeness of multiply warped space–times by using similar arguments in [16]. First, we will state the following result which is an extension of Lemma 3.1 of [16].

Lemma 4.5. *$M = (c, d) \times_{b_1} F_1 \times \cdots \times_{b_m} F_m$ be a Lorentzian multiply warped product with metric $g = -dt^2 \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$, where $-\infty \leq c < d \leq \infty$, also let $\gamma = (\alpha, \beta_1, \ldots, \beta_m)\colon [0, \delta) \to M$ be a geodesic for some $\delta > 0$. If (F_i, g_{F_i}) is complete for any $i \in \{1, \ldots, m\}$, then the following conditions are equivalent*
1. *γ is extendible as a geodesic past δ.*
2. *α is continuously extendible to δ.*
3. *$\alpha'[0, \delta)$ is in a compact subset of T(B).*
4. *$\alpha[0, \delta)$ is in a compact subset of B.*

We will express the length of α', i.e., $||\alpha'||$ in terms of D and c_i, where $\gamma = (\alpha, \beta_1, \ldots, \beta_m)$, $g(\gamma', \gamma') = D$ and $(b_i \circ \alpha)^4 g_{F_i}(\beta_i', \beta_i') = c_i$, for any $i \in \{1, \ldots, m\}$.

Lemma 4.6. *$M = (c, d) \times_{b_1} F_1 \times \cdots \times_{b_m} F_m$ be a Lorentzian multiply warped product with metric $g = -dt^2 \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$, where $-\infty \leq c < d \leq \infty$, also let $\gamma = (\alpha, \beta_1, \ldots, \beta_m)\colon I \to M$ be a future directed geodesic. Suppose that the speed of γ is D, (i.e., $g(\gamma', \gamma') = D$).*
1. *If $\lim_{t\to d^-} \int_{w_0}^t (-D + \sum_{i=1}^m c_i/b_i^2(s))^{-1/2}\, \mathrm{d}s = \infty$ for some $w_0 \in (c, d)$, then γ is a future complete geodesic.*
2. *If $\lim_{t\to c^+} \int_t^{w_0} (-D + \sum_{i=1}^m c_i/b_i^2(s))^{-1/2}\, \mathrm{d}s = \infty$ for some $w_0 \in (c, d)$, then γ is a past complete geodesic.*

Proof. We have $g(\gamma', \gamma') = D$, i.e., $D = -(\alpha')^2 + \sum_{i=1}^{m}(b_i \circ \alpha)^2 g_{F_i}(\beta_i', \beta_i')$. By using Remark 2.5 we obtain,

$$D\prod_{i=1}^{m}(b_i \circ \alpha)^2 = (\alpha')^2\prod_{i=1}^{m}(b_i \circ \alpha)^2 + \left(\sum_{i=1}^{m}(b_i \circ \alpha)^2 g_{F_i}(\beta_i', \beta_i')\right)\prod_{i=1}^{m}(b_i \circ \alpha)^2$$
$$= -(\alpha')^2\prod_{i=1}^{m}(b_i \circ \alpha)^2 + \sum_{i=1}^{m}(b_i \circ \alpha)^2 \cdots (b_{i-1} \circ \alpha)^2 c_i (b_{i+1} \circ \alpha)^2 \cdots (b_m \circ \alpha)^2.$$

Hence,

$$\|\alpha'(s)\| = \left(-D + \sum_{i=1}^{m}\frac{c_i}{b_i^2(s)}\right)^{1/2} \tag{2}$$

Hence, the result follows from Lemmas 3.5 and 3.8 of [16] and Lemma 4.5. □

Now, we will introduce the following notation which will be useful in stating a number of results.

Notation. $M = (c, d) \times_{b_1} F_1 \times \cdots \times_{b_m} F_m$ be a Lorentzian multiply warped product with metric $g = -dt^2 \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$, where $-\infty \le c < d \le \infty$. If $\mathcal{B} = \{b_1, \dots, b_m\}$ and for some $k \in \{1, \dots, m\}$ and for some subset $\{\bar{b}_1, \dots, \bar{b}_k\}$ of $\mathcal{B}$, then

$$f[\bar{b}_1, \dots, \bar{b}_k] = \prod_{i=1}^{k}\bar{b}_i \quad \text{and} \quad h[\bar{b}_1, \dots, \bar{b}_k] = \sum_{i=1}^{k}\bar{b}_1^2 \cdots \bar{b}_{i-1}^2 \bar{b}_{i+1}^2 \cdots \bar{b}_k^2.$$

Also, it is assumed that $h[\bar{b}_1] = 1$, for any $\bar{b}_1 \in \mathcal{B}$.

By making use of Lemmas 3.5 and 3.8 of [16] and Lemma 4.6 and also considering three different cases, i.e., $D = 0$ for null, $D = -1$ for time-like and $D = 1$ for space-like geodesics we obtain the following results.

Theorem 4.7. *$M = (c, d) \times_{b_1} F_1 \times \cdots \times_{b_m} F_m$ be a Lorentzian multiply warped product with metric $g = -dt^2 \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$, where $-\infty \le c < d \le \infty$. Suppose that (F_i, g_{F_i}) is complete for any $i \in \{1, \dots, m\}$ and $\mathcal{B} = \{b_1, \dots, b_m\}$ then*

1. $\lim_{t \to d^-} \int_{w_0}^{t} f[\bar{b}_1, \dots, \bar{b}_k](s)/\sqrt{h[\bar{b}_1, \cdots, \bar{b}_k](s)}\,\mathrm{d}s = \infty$ *for some $w_0 \in (c, d)$ and for any $k \in \{1, \dots, m\}$ and any subset $\{\bar{b}_1, \dots, \bar{b}_k\}$ of $\mathcal{B}$ if and only if every future directed null geodesic is future complete.*
2. $\lim_{t \to c^+} \int_{t}^{w_0} f[\bar{b}_1, \dots, \bar{b}_k](s)/\sqrt{h[\bar{b}_1, \dots, \bar{b}_k](s)}\,\mathrm{d}s = \infty$ *for some $w_0 \in (c, d)$ and for any $k \in \{1, \dots, m\}$ and any subset $\{\bar{b}_1, \dots, \bar{b}_k\}$* of $\mathcal{B}$ *if and only if every future directed null geodesic is past complete.*

Theorem 4.8. *$M = (c, d) \times_{b_1} F_1 \times \cdots \times_{b_m} F_m$ be a Lorentzian multiply warped product with metric $g = -dt^2 \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$, where $-\infty \le c < d \le \infty$. Suppose that (F_i, g_{F_i}) is complete for any $i \in \{1, \dots, m\}$ and $\mathcal{B} = \{b_1, \dots, b_m\}$ then*

1. $\lim_{t\to d^-}\int_{w_0}^{t} f[\bar{b}_1,\ldots,\bar{b}_k](s)/\sqrt{f[\bar{b}_1,\ldots,\bar{b}_k]^2(s)+h[\bar{b}_1,\ldots,\bar{b}_k](s)}\,\mathrm{d}s = \infty$ *for some* $w_0 \in (c,d)$ *and for any* $k \in \{1,\ldots,m\}$ *and any subset* $\{\bar{b}_1,\ldots,\bar{b}_k\}$ *of* $\mathcal{B}$ *if and only if every future directed time-like geodesic is future complete.*
2. $\lim_{t\to c^+}\int_{t}^{w_0} f[\bar{b}_1,\ldots,\bar{b}_k](s)/\sqrt{f[\bar{b}_1,\ldots,\bar{b}_k]^2(s)+h[\bar{b}_1,\ldots,\bar{b}_k](s)}\,\mathrm{d}s = \infty$ *for some* $w_0 \in (c,d)$ *and for any* $k \in \{1,\ldots,m\}$ *and any subset* $\{\bar{b}_1,\ldots,\bar{b}_k\}$ *of* $\mathcal{B}$ *if and only if every future directed time-like geodesic is past complete.*

Theorem 4.9. $M = (c,d)\times_{b_1} F_1 \times\cdots\times_{b_m} F_m$ *be a Lorentzian multiply warped product with metric* $g = -dt^2 \oplus b_1^2 g_{F_1} \oplus\cdots\oplus b_m^2 g_{F_m}$, *where* $-\infty \le c < d \le \infty$. *Suppose that* (F_i, g_{F_i}) *is complete for any* $i \in \{1,\ldots,m\}$ *and* $\mathcal{B} = \{b_1,\ldots,b_m\}$ *then*
1. *Either* $\lim_{t\to d^-}\int_{w_0}^{t} f[\bar{b}_1,\ldots,\bar{b}_k](s)/\sqrt{h[\bar{b}_1,\ldots,\bar{b}_k](s)}\,\mathrm{d}s = \infty$ *or* $f[\bar{b}_1,\ldots,\bar{b}_k](s)/\sqrt{h[\bar{b}_1,\ldots,\bar{b}_k](s)}$ *is an unbounded function on* (w_0,d) *for some* $w_0 \in (c,d)$ *and for any* $k \in \{1,\ldots,m\}$ *and any subset* $\{\bar{b}_1,\ldots,\bar{b}_k\}$ *of* $\mathcal{B}$ *if any only if every future directed space-like geodesic is future complete.*
2. *Either* $\lim_{t\to c^+}\int_{t}^{w_0} f[\bar{b}_1,\ldots,\bar{b}_k](s)/\sqrt{h[\bar{b}_1,\ldots,\bar{b}_k](s)}\,\mathrm{d}s = \infty$ *or* $f[\bar{b}_1,\ldots,\bar{b}_k](s)/\sqrt{h[\bar{b}_1,\ldots,\bar{b}_k](s)}$ *is an unbounded function on* (c,w_0) *for some* $w_0 \in (c,d)$ *and for any* $k \in \{1,\ldots,m\}$ *and any subset* $\{\bar{b}_1,\ldots,\bar{b}_k\}$ *of* $\mathcal{B}$ *if any only if every future directed space-like geodesic is past complete.*

Corollary 4.10. $M = (c,d)\times_{b_1} F_1 \times\cdots\times_{b_m} F_m$ *be a Lorentzian multiply warped product with metric* $g = -dt^2 \oplus b_1^2 g_{F_1} \oplus\cdots\oplus b_m^2 g_{F_m}$, *where* $-\infty \le c < d \le \infty$. *Suppose that* (F_i, g_{F_i}) *is complete for any* $i \in \{1,\ldots,m\}$ *then*
1. *If* (M,g) *is time-like complete, then* (M,g) *is null complete.*
2. *If* $0 < \inf(b_i) < \sup(b_i) < 0$, *for any* $i \in \{1,\ldots,m\}$, *then* (M,g) *is null complete iff* (M,g) *is time-like complete.*

Proof. These follow from Theorems 4.7 and 4.8. □

Lemma 4.11. *Let* $M = B\times_{b_1} F_1 \times\cdots\times_{b_m} F_m$ *be a pseudo-Riemannian multiply warped product with metric* $g = g_B \oplus b_1^2 g_{F_1} \oplus\cdots\oplus b_m^2 g_{F_m}$ *and also let* $\beta_i\colon I \to F_i$ *be a unit speed geodesic of* F_i *for some* $i \in \{1,\ldots,m\}$ *and* $\gamma = (\alpha,\beta_1,\ldots,\beta_m)\colon I \to M$ *be a curve in M. If there exist points* $q_j \in F_j$, *for all* $j \in \{1,\ldots,m\} - \{i\}$ *such that* $\beta_j(s) = q_j$, *for any* $s \in I$, *then* $\gamma = (\alpha,\beta_1,\ldots,\beta_m)$ *is a pre-geodesic in M if and only if*

$$\nabla^B_{\alpha'}\alpha'(s) - (b_i\circ\alpha)(s)\,\mathrm{grad}_B(b_i)(\alpha(s)) = 2\frac{(\alpha' b_i)(s)}{(b_i\circ\alpha)(s)}\alpha'(s)$$

for all $s \in I$.

Proof. Note that $\gamma = (\alpha,\beta_1,\ldots,\beta_m)$ is a pre-geodesic in M if and only if there exists a smooth function $h\colon I \to \mathbb{R}$ such that $\nabla_{\gamma'}\gamma'(s) = h(s)\gamma'(s)$, for all $s \in I$. It is clear that $\beta_j'(s) = 0$ for all $j \in \{1,\ldots,m\} - \{i\}$ and for any $s \in I$. By using the *covariant derivative formulas* we get

$$\nabla_{\gamma'}\gamma'(s) = \nabla^B_{\alpha'}\alpha'(s) - (b_i\circ\alpha)(s)\,\mathrm{grad}_B(b_i)(\alpha(s)) + 2\frac{(\alpha' b_i)(s)}{(b_i\circ\alpha)(s)}\beta_i'(s)$$

By projecting onto $T_{\beta_i(s)}(F_i)$ we have $2(\alpha' b_i)(s)/(b_i \circ \alpha)(s)\beta_i'(s) = h(s)\beta_i'(s)$ this implies that $h(s) = 2(\alpha' b_i)(s)/(b_i \circ \alpha)(s)$. Therefore the result follows by projecting the above *covariant derivative formula* onto $T_{\alpha(s)}(B)$. □

Theorem 4.12. *Let $M = B \times_{b_1} F_1 \times \cdots \times_{b_m} F_m$ be a Lorentzian multiply warped product with metric $g = g_B \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$. If (M, g) is null, time-like or space-like complete, then (F_i, g_{F_i}) is a complete Riemannian manifold for any $i \in \{1, \ldots, m\}$.*

Proof. (This result was first stated for singly Lorentzian warped products in [14] and an alternative proof for singly Lorentzian warped products was given in [22].) Let us fix $p \in B$ and let $X_p \in T_p(B)$ be a time-like with $g_B(X_p, X_p) = -b_i^2(p)$, for some $i \in \{1, \ldots, m\}$. Now we can show that one can find a smooth curve α: $[0, \epsilon) \to B$, where $\epsilon > 0$ such that

$$\nabla^B_{\alpha'}\alpha'(s) - (b_i \circ \alpha)(s)\,\mathrm{grad}_B(b_i)(\alpha(s)) = 2\frac{(\alpha' b_i)(s)}{(b_i \circ \alpha)(s)}\alpha'(s), \qquad \alpha'(0) = X_p. \quad (3)$$

System (3) is an initial value problem then clearly the existence and uniqueness theorem for ordinary differential equations gives the following result: there exists a $\epsilon > 0$ and a differentiable curve α: $[0, \epsilon) \to B$ satisfying system (3). Let $\tilde{\beta}_i$: $[0, K) \to F_i$ be a unit speed maximally extended geodesic and $K < \infty$. Let $L = K - \epsilon/2$ and set $\beta_i = \tilde{\beta}_i(t + L)$ so β_i: $[-L, \epsilon/2) \to F_i$. Thus β_i is a unit speed geodesic and can be extended to $t = \epsilon/2$. Set $\gamma(t) = (\alpha(t), \beta_1(t), \ldots, \beta_m(t))$, where $\beta_j(t) = q_j$, for some points $q_j \in F_j$, for all $j \in \{1, \ldots, m\} - \{i\}$ and any $t \in I$, by Lemma 4.11, we have γ is a pre-geodesic with

$$\nabla_{\gamma'}\gamma'(t) = h(t)(\alpha'(t) + \beta_i'(t)), \quad \text{where } h(t) = 2\frac{(\alpha' b_i)(t)}{(b_i \circ \alpha)}.$$

But $2(\alpha' b_i)(t)/(b_i \circ \alpha)$ must be bounded and smooth on the closed interval $[0, \epsilon/2]$, hence $h(t)$ must be bounded and smooth on the closed interval $[0, \epsilon/2)$. By converting to an affine parameter (cf. [20]) s via p, we get

$$s = p(t) = \int_0^t (\mathrm{e}^{\int 0u h(v)\,\mathrm{d}v})\,\mathrm{d}u.$$

Notice that $s = 0$, i.e., $t = 0$ corresponds to $\gamma(0) = (\alpha(0), \beta_1(0), \ldots, \beta_m(0)) = (p, \tilde{\beta}_1(0), \ldots, \tilde{\beta}_m(0))$ and $\gamma'(0) = \alpha'(0) + \beta_i'(0)$. Also

$$g(\gamma'(0), \gamma'(0)) = g_B(\alpha'(0), \alpha'(0)) + b_i^2(p) g_F(\beta_i'(0), \beta_i'(0)) = -b_i^2(p) + b_i^2(p) = 0.$$

Hence γ is null pre-geodesic of M. The affine parameter

$$s_0 = \lim_{t \to \epsilon/2} p(t) = \lim_{t \to \epsilon/2} \int_0^t (\mathrm{e}^{\int 0u h(v)\,\mathrm{d}v})\,\mathrm{d}u$$

must be finite, $s_0 < \infty$. Because h is bounded on $[0, \epsilon/2)$. Since we cannot extend β_i to $t = \epsilon/2$ then we cannot extend γ to $t = \epsilon/2$. This implies that we cannot extend $\tilde{\gamma}(s) = \gamma \circ p^{-1}(s) = (\alpha(t(s)), \beta_1(t(s)), \ldots, \beta_m(t(s)))$ to $s = s_0$. Thus the geodesic $\tilde{\gamma} = \gamma \circ p^{-1}$ is not complete. □

In contrast to the above showing (F, g_F) is complete, the corresponding for (B, g_B) is much simpler.

Proposition 4.13. *Let $M = B \times_{b_1} F_1 \times \cdots \times_{b_m} F_m$ be a Lorentzian multiply warped product with metric $g = g_B \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$. If (M, g) is null, time-like or space-like complete, then (B, g_B) is either null, time-like or a space-like complete Lorentzian manifold, respectively.*

Proof. Let α be an incomplete time-like (respectively, null, space-like) geodesic in B. Then for any $q \in F$, $\gamma = (\alpha, q)$ is an incomplete time-like (respectively, null, space-like) geodesic of M by Proposition 2.4. □

4.2. Riemannian warped products

In this section, we state some results about completeness of Riemannian multiply warped products.

Theorem 4.14. *Let $M = B \times_{b_1} F_1 \times \cdots \times_{b_m} F_m$ be a Riemannian multiply warped product with metric $g = g_B \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$. If (B, g_B) and (F_i, g_{F_i}) are all complete Riemannian manifolds for any $i \in \{1, \ldots, m\}$ then (M, g) is a complete Riemannian manifold.*

Proof. We use the metric completeness criterion from the Hopf–Rinow theorem in 5.21 [13]. Note first that if X is tangent to M, i.e., $X \in \mathcal{L}(B)$ then $g(X, X) = g_B(d\pi(X), d\pi(X))$. Hence, $L(\gamma) \geq L(\alpha)$ for any curve segment $\gamma = (\alpha, \beta_1, \ldots, \beta_m)$; hence $d((p, q), (p', q')) \geq d_B(p, p')$ for all $(p, q), (p', q') \in M$. This property implies that, if (p_n, q_n) is a Cauchy sequence in M, then (p_n) is Cauchy in B. Since (B, g_B) is complete, p_n converges to some point $p \in B$. We can assume that the sequence lies in some compact set P in B; hence $b_i \geq k_i > 0$ on P for some $k_i > 0$, and any $i \in \{1, \ldots, m\}$. Then a variant of the argument shows that $d((p, q), (p', q')) \geq \min\{k_i\} \sum_{i=1}^{m} d_{F_i}(q, q')$ for all $(p, q), (p', q') \in P \times F$. Now (q_n) is Cauchy in F and thus converges; so the original sequence converges and M is complete. □

Theorem 4.15. *Let $M = B \times_{b_1} F_1 \times \cdots \times_{b_m} F_m$ be a Riemannian multiply warped product with metric $g = g_B \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$. If (M, g) is a complete Riemannian manifold, then (B, g_B) and (F_i, g_{F_i}) are complete Riemannian manifolds for any $i \in \{1, \ldots, m\}$.*

Proof. Let (p_n) be Cauchy in B. Then for a fixed $q \in F$ we have (p_n, q) is Cauchy in M. Because $d((p_n, q), (p_m, q)) = d_B(p_n, p_m)$. Thus there is a point $(p, q) \in M$ such that $\lim(p_n, q) = (p, q)$. Then since $d((p_n, q), (p, q)) = d_B(p_n, p)$ we have $\lim(p_n) = p$. Hence B is complete. Now, suppose that q_n^i is Cauchy in F_i for an arbitrary $i \in \{1, \ldots, m\}$ and q_j are point in F_i for any $j \in \{1, \ldots, m\} - \{i\}$, respectively. Then for a fixed points $p \in B$, we have that $(p, q_n^i) = (p, q_1, \ldots, q_{i-1}, q_n^i, q_{i+1}, \ldots, q_m)$ is Cauchy in M since

$d((p,q_n^i),(p,q_m^i)) = b_i(p)d_{F_i}(q_n^i,q_m^i)$. Thus $\lim(p,q_n^i) = (p,q)$ exists, where $(p,q) = (p,q_1,\ldots,q_{i-1},q_i,q_{i+1},\ldots,q_m)$ and $q_i \in F_i$. This implies that $d((p,q_n^i),(p,q)) = b_i(p)d_{F_i}(q_n^i,q_i)$. Therefore, $\lim q_i^n = q_i$ and F_i is complete for some i, hence for any $i \in \{1,\ldots,m\}$. □

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